Lower Bound for Derivatives of Costa's Differential Entropy

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***Abstract*—Let $H(X_t)$ be the differential entropy of an n-dimensional random vector X_t introduced by Costa. Cheng and Geng conjectured that $C_1(m,n): (-1)^{m+1}(\mathrm{d}^m/\mathrm{d}^m t)H(X_t) \geq 0$. McKean conjectured that $C_2(m,n): (-1)^{m+1}(\mathrm{d}^m/\mathrm{d}^m t)H(X_t) \geq (-1)^{m+1}(\mathrm{d}^m/\mathrm{d}^m t)H(X_{Gt})$. McKean's conjecture was only considered in the univariate case before: $C_2(1,1)$ and $C_2(2,1)$ were proved by McKean and $C_2(i,1), i = 3,4,5$ were proved by Zhang-Anantharam-Geng under the log-concave condition. In this paper, we prove $C_2(1,n)$, $C_2(2,n)$ and observe that McKean's conjecture might not be true for $n > 1$ and $m > 2$. We further propose a weaker conjecture $C_3(m,n)$: $(-1)^{m+1}(\mathrm{d}^m/\mathrm{d}^m t)H(X_t) \geq (-1)^{m+1}\frac{1}{n}(\mathrm{d}^m/\mathrm{d}^m t)H(X_{Gt})$ and prove $C_3(3,2)$, $C_3(3,3)$, $C_3(3,4)$ under the log-concave condition. A systematic procedure to prove $C_l(m,n)$ is proposed and the results mentioned above are proved using this procedure.**

I. Introduction

Shannon's *entropy power inequality (EPI)* is one of the most important information inequalities [1], which has many proofs, generalizations, and applications [2]–[10]. In particular, Costa presented a stronger version of the EPI [11].

Let X be an n-dimensional random vector with *probability density* $p(x)$. For $t > 0$, define $X_t \triangleq X + Z_t$, where $Z_t \sim N_n(0, tI)$ is independent of X. Let $p_t(x_t)$ be the *probability density* of X_t [19]. *Costa's differential entropy* is defined to be

$$H(X_t) = -\int_{\mathbb{R}^n} p_t(x_t)\log p_t(x_t)\mathrm{d}x_t. \tag{1}$$

Costa [11] proved that the *entropy power* of X_t satisfies $(\mathrm{d}/\mathrm{d}t)N(X_t) \geq 0$ and $(\mathrm{d}^2/\mathrm{d}^2 t)N(X_t) \leq 0$. Several new proofs and generalizations for Costa's EPI were given [12], [15], [16]. Cheng and Geng further proposed a conjecture [14]:

Conjecture 1. $H(X_t)$ is *completely monotone* in t, that is, $C_1(m,n): (-1)^{m+1}(\mathrm{d}^m/\mathrm{d}^m t)H(X_t) \geq 0$.

Costa's EPI implies $C_1(1,n)$ and $C_1(2,n)$ [11], Cheng-Geng proved $C_1(3,1)$ and $C_1(4,1)$ [14]. In [18], $C_1(3,2)$, $C_1(3,3)$, $C_1(3,4)$ were proved with a systematic procedure.

Let $X_G \sim N_n(\mu, \sigma^2 I)$ be an n-dimensional Gaussian random vector and $X_{Gt} \triangleq X_G + Z_t$. McKean [17] proved that X_{Gt} achieves the minimum of $(\mathrm{d}/\mathrm{d}t)H(X_t)$ and $-(\mathrm{d}^2/\mathrm{d}^2 t)$ $H(X_t)$ subject to $\mathrm{Var}(X_t) = \sigma^2 + t$, and conjectured

Conjecture 2. Subject to $\mathrm{Var}(X_t) = \sigma^2 + t$, we have $C_2(m,n): (-1)^{m+1}(\mathrm{d}^m/\mathrm{d}^m t)H(X_t) \geq (-1)^{m+1}(\mathrm{d}^m/\mathrm{d}^m t)H(X_{Gt})$.

McKean proved $C_2(1,1)$ and $C_2(2,1)$ [17]. Zhang-Anantharam-Geng [13] proved $C_2(3,1)$, $C_2(4,1)$, and $C_2(5,1)$ if the probability density function of X_t is log-concave.

In this paper, we verified that $C_2(3,n), n > 1$ could not be proved by our procedure. So we conjecture that $C_2(3,n), n > 1$ may not be true and propose

Conjecture 3. Subject to $\mathrm{Var}(X_t) = \sigma^2 + t$, we have $C_3(m,n): (-1)^{m+1}(\mathrm{d}^m/\mathrm{d}^m t)H(X_t) \geq (-1)^{m+1}\frac{1}{n}(\mathrm{d}^m/\mathrm{d}^m t)H(X_{Gt})$.

Conjecture 2 implies Conjecture 3 and Conjecture 3 implies Conjecture 1, since $H(X_{Gt}) \geq 0$ [13].

In this paper, we propose a systematic and effective procedure to prove $C_s(m,n)$, which consists of three main ingredients. First, a systematic method is proposed to compute constraints $R_i, i = 1,\ldots,N_1$ satisfied by $p_t(x_t)$ and its derivatives. The condition that p_t is log-concave can also be reduced to a set of constraints $\mathcal{R}_j, j = 1,\ldots,N_2$. Second, proof for $C_s(m,n)$ is reduced to the following problem

$$\exists p_i \in \mathbb{R} \text{ and } Q_j \text{ s.t. } (E - \sum_{i=1}^{N_1} p_i R_i - \sum_{j=1}^{N_2} Q_j \mathcal{R}_j = S) \tag{2}$$

where Q_j is a polynomial in p_t and its derivatives such that $Q_j \geq 0$ and S is a sum of squares (SOS). Third, problem (2) can be solved with the semidefinite programming (SDP) [20], [21]. There exists no guarantee that the procedure will generate a proof, but when succeeds, it gives a strict proof for $C_s(m,n)$.

Using the above procedure, we first prove $C_2(1,n)$, $C_2(2,n)$. Then we prove $C_3(3,2)$, $C_3(3,3)$ and $C_3(3,4)$ under the condition that p_t is log-concave. $C_2(3,2)$, $C_2(3,3)$ and $C_2(3,4)$ cannot be proved with the above procedure even if p_t is log-concave, which motivates us to propose Conjecture 3.

	$C_2(3,1)$	$C_3(3,2)$	$C_3(3,3)$	$C_3(3,4)$	$C_2(2,n)$
Vars	3	14	38	38	6
N_1	6	63	512	512	8
N_2	0	0	6	6	0
Time	0.18	0.53	9.00	9.02	0.32

TABLE I
Data in computing the SOS with SDP

In Table I, we give the data for computing (2), where Vars is the number of variables (seen in Procedure 2.6), and Time is the running time in seconds collected on a desktop PC with a 3.40GHz CPU.

The procedure is inspired by the work [11]–[14], and uses basic ideas introduced therein. In particular, our approach can be basically considered as a generalization of [13] from the univariate case to the multivariate case and as a generalization of [18] by adding the log-concave constraints. We further remark that it is not straightforward to extend the method from the univariate case to the multivariate case which makes the computational complexity increase dramatically. So as our method, it is necessary to combine symbolic computation and semidefinite programming to reduce the complexity. Also, compared to [13], the log-concave constraints considered in this paper are more general, and using SOS in this paper gives an explicit proof.

II. Proof Procedure

In this section, we give a general procedure to prove $C_s(m,n)$ for specific values of s, m, n.

A. Notations

Let $[n] = \{1,\ldots,n\}$, $[n]_0 = \{0,1,\ldots,n\}$, and $x_t = [x_{1,t},\ldots,x_{n,t}]$. We use p_t to denote $p_t(x_t)$ and denote

$$\mathcal{P}_n = \{\frac{\partial^h p_t}{\partial^{h_1} x_{1,t}\cdots\partial^{h_n} x_{n,t}} : h = \sum_{i=1}^{n} h_i, h_i \in \mathbb{N}\}$$

and $\mathbb{R}[\mathcal{P}_n]$ to be the set of polynomials in $\mathcal{P}_n$. For $v \in \mathcal{P}_n$, let $\mathrm{ord}(v)$ be the order of v. For a monomial $\prod_{i=1}^{r} v_i^{d_i}$ with $v_i \in \mathcal{P}_n$, its *degree*, *order*, and *total order* are defined to be $\sum_{i=1}^{r} d_i$, $\max_{i=1}^{r} \mathrm{ord}(v_i)$, and $\sum_{i=1}^{r} d_i \cdot \mathrm{ord}(v_i)$, respectively.

A polynomial in $\mathbb{R}[\mathcal{P}_n]$ is called a kth-order *differential form*, if all its monomials have degree k and total order k. Let $\mathcal{M}_{k,n}$ be the set of all monomials with degree k and total order k. Then the set of kth-order differential forms is an $\mathbb{R}$-linear vector space generated by $\mathcal{M}_{k,n}$, which is denoted as $\mathrm{Span}_{\mathbb{R}}(\mathcal{M}_{k,n})$.

We will use Gaussian elimination in $\mathrm{Span}_{\mathbb{R}}(\mathcal{M}_{k,n})$ by treating the monomials as bases. We always use the *lexicographic order for the monomials* as defined in [19].

B. The proof procedure for $C_s(m,n)$

The proof procedure consists of four steps.

In step 1, we reduce the proof of $C_s(m,n)$ into the proof of an integral inequality, as shown by the following lemma, whose proof will be given in section II-C.

Lemma 2.1: Proof of $C_s(m,n), s = 1,2,3$ can be reduced to show

$$\int_{\mathbb{R}^n} E_{s,m,n}/p_t^{2m-1}\mathrm{d}x_t \geq 0 \tag{3}$$

where $E_{s,m,n} = \sum_{a_1=1}^{n}\cdots\sum_{a_m=1}^{n} E_{s,m,n,\mathbf{a}_m}$, $\mathbf{a}_m = (a_1,\ldots,a_m)$, $E_{s,m,n,\mathbf{a}_m}$ is a $2m$th-order differential form in $\mathbb{R}[\mathcal{P}_{m,n}]$, and

$$\mathcal{P}_{m,n} = \{\frac{\partial^h p_t}{\partial^{h_1} x_{a_1,t}\cdots\partial^{h_m} x_{a_m,t}} : h \in [2m-1]_0; a_i \in [n], i \in [m]\}. \tag{4}$$

In step 2, we compute the constraints satisfied by p_t. We consider two types of constraints: integral constraints and log-concave constraints which will be given in Lemmas 2.3 and 2.5, respectively. Since $E_{s,m,n}$ in (3) is a $2m$th-order differential form, we only consider constraints which are $2m$th-order differential forms.

Definition 2.2: An mth-order *integral constraint* is a $2m$th-order differential form R in $\mathbb{R}[\mathcal{P}_n]$ s.t. $\int_{\mathbb{R}^n} \frac{R}{p_t^{2m-1}}\mathrm{d}x_t = 0$.

Lemma 2.3 ([18]): There is a method to compute the mth-order integral constraints $\mathcal{C}_{m,n} = \{R_i, i = 1,\ldots,N_1\}$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *log-concave* if $\log f$ is a concave function. In this paper, by the *log-concave condition*, we mean that the density function p_t is log-concave.

Definition 2.4: An mth-order *log-concave constraint* is a $2m$th-order differential form $\mathcal{R}$ in $\mathbb{R}[\mathcal{P}_n]$ such that $\mathcal{R} \geq 0$ under the log-concave condition.

The following lemma computes the log-concave constraints, whose proof is given in section II-D.

Lemma 2.5: Let $\mathbf{H}(p_t) \in \mathbb{R}[\mathcal{P}_n]^{n\times n}$ be the Hessian matrix of p_t, $\nabla p_t = (\frac{\partial p_t}{\partial x_{1,t}},\ldots,\frac{\partial p_t}{\partial x_{n,t}})$,

$$\mathbf{L}(p_t) \triangleq p_t\mathbf{H}(p_t) - \nabla^T p_t \nabla p_t, \tag{5}$$

and $\triangle_{k,l}, l = 1,\ldots,L_k$ the kth-order principle minors of $\mathbf{L}(p_t)$. Then the mth-order log-concave constraints are

$$\mathbf{C}_{m,n} = \{\prod_{i=1}^{l}(-1)^{k_i}\triangle_{k_i,l_i}T_{k_1,\ldots,k_l} \mid \sum_{i=1}^{l} k_i \leq m\} \tag{6}$$

where $T_{k_1,\ldots,k_l} \in \mathrm{Span}_{\mathbb{R}}(\mathcal{M}_{2m-2\sum_{i=1}^{l}k_i,n})$ and $T_{k_1,\ldots,k_l} \geq 0$.

For convenience, denote

$$\mathbb{C}_{m,n} = \{P_j, j = 1,\ldots,N_2\}, \tag{7}$$

where P_j represents $\prod_{i=1}^{l}(-1)^{k_i}\triangle_{k_i,l_i}$ in (6), which is a $(2\sum_{i=1}^{l}k_i)$th-order log-concave constraint by Lemma 2.5.

In step 3, we give a procedure to write $E_{s,m,n}$ as an SOS, detail of which will be given in section II-E.

Procedure 2.6: For $E_{s,m,n}$ in Lemma 2.1, $\mathcal{C}_{m,n} = \{R_i, i = 1,\ldots,N_1\}$ in Lemma 2.3, and $\mathbb{C}_{m,n} = \{P_j, j = 1,\ldots,N_2\}$ in Lemma 2.5, the procedure computes $e_i \in \mathbb{R}$ and $Q_j \in \mathrm{Span}_{\mathbb{R}}(\mathcal{M}_{2m-\deg P_j,n})$ such that

$$E_{s,m,n} - \sum_{i=1}^{N_1} e_iR_i - \sum_{j=1}^{N_2} P_jQ_j = S \quad \text{and} \tag{8}$$

$$Q_j \geq 0, j = 1,\ldots,N_2 \tag{9}$$

where S is an SOS. If the log-concave condition is not needed, we may set $Q_j = 0$ for all j.

To summarize, we have

Theorem 2.7: If Procedure 2.6 finds (8) and (9) for certain s, m, n, then $C_s(m,n)$ is true.

Proof: By Lemma 2.1, we have a proof for $C_s(m,n)$:

$$\begin{aligned}\int_{\mathbb{R}} \frac{E_{t,m,n}}{p_t^{2m-1}}\mathrm{d}x_t &\overset{(8)}{=} \int_{\mathbb{R}} \frac{\sum_{i=1}^{N_1} e_iR_i + \sum_{j=1}^{N_2} P_jQ_j + S}{p_t^{2m-1}}\mathrm{d}x_t \\ &\overset{S1}{=} \int_{\mathbb{R}} \frac{\sum_{j=1}^{N_2} P_jQ_j + S}{p_t^{2m-1}}\mathrm{d}x_t \overset{S2}{\geq} \int_{\mathbb{R}} \frac{S}{p_t^{2m-1}}\mathrm{d}x_t \overset{S3}{\geq} 0.\end{aligned} \tag{10}$$

Equality S1 is true, because R_i is an integral constraint by Lemma 2.3. By Lemma 2.5 and (9), $P_jQ_j \geq 0$ is true under the log-concave condition, so inequality S2 is true under the log-concave condition. Finally, inequality S3 is true, because $S \geq 0$ is an SOS. ■

C. Proof of Lemma 2.1

Costa [11] proved the following basic properties

$$\begin{aligned}&\frac{\mathrm{d}p_t}{\mathrm{d}t} = \frac{1}{2}\nabla^2 p_t \\ &\frac{\mathrm{d}H(X_t)}{\mathrm{d}t} = -\frac{1}{2}\mathbb{E}[\nabla^2 \log p_t] = \frac{1}{2}\int_{\mathbb{R}^n} \frac{\|\nabla p_t\|^2}{p_t}\mathrm{d}x_t = \frac{1}{2}J(X_t),\end{aligned} \tag{11}$$

where $\nabla^2 p_t = \sum_{i=1}^{n} \frac{\partial^2 p_t}{\partial^2 x_{i,t}}$ and $J(X_t) \triangleq \mathbb{E}\left(\frac{\|\nabla p_t\|^2}{p_t^2}\right)$ is the *Fisher information* [6]. p_t satisfies the *heat equation* by equation (11).

For $s = 1$, Lemma 2.1 was proved in [18]:

Lemma 2.8 ([18]): For $m \in \mathbb{N}_{m>1}$, we have

$$(-1)^{m+1}(\mathrm{d}^m/\mathrm{d}^m t)H(X_t) = \int_{\mathbb{R}^n} E_{1,m,n}/p_t^{2m-1}(x_t)\mathrm{d}x_t, \tag{12}$$

where $E_{1,m,n} = p_t^{2m-1}[(-1)^{m+1}\frac{1}{2}\frac{\mathrm{d}^{m-1}}{\mathrm{d}^{m-1}t}(\frac{\|\nabla p_t\|^2}{p_t}) = \sum_{a_1=1}^{n}\cdots\sum_{a_m=1}^{n} E_{1,m,n,\mathbf{a}_m}$ is a $2m$th-order differential form in $\mathbb{R}[\mathcal{P}_{m,n}]$.

To prove Lemma 2.1 for $s = 2,3$, we need to compute $(\mathrm{d}^m/\mathrm{d}^m t)H(X_{Gt})$. Since $X_G \sim N_n(\mu,\sigma^2 I)$ and $X_{Gt} \triangleq X_G + Z_t$, $X_{Gt} \sim N_n(\mu,(\sigma^2+t)I)$ and the probability density of X_{Gt} is $\widehat{p}_t = \frac{1}{(2\pi(\sigma^2+t))^{n/2}}\exp(-\frac{1}{2(\sigma^2+t)}\|x_t-\mu\|^2)$.

Lemma 2.9 ([19]): Let $T = \nabla^2 \log p_t$ and $T_G = \nabla^2 \log \widehat{p}_t$. Then under the log-concave condition, we have

$$\mathbb{E}[(-T)^m] \overset{(a)}{\geq} [\mathbb{E}(-T)]^m \overset{(b)}{\geq} [\mathbb{E}(-T_G)]^m \overset{(c)}{=} (-1)^{m+1} \frac{2n^{m-1}}{(m-1)!} (\mathrm{d}^m/\mathrm{d}^m t) H(X_{Gt}). \tag{13}$$

Lemma 2.10 ([19]): For $T = \nabla^2 \log p_t$ and $m \in \mathbb{N}_{m>1}$, we have

$$\mathbb{E}[(-T)^m] = \int_{\mathbb{R}}^{n} \frac{E_{0,m,n}}{p_t^{2m-1}} dx_t \tag{14}$$

where $E_{0,m,n} = \sum_{a_1=1}^{n} \cdots \sum_{a_m=1}^{n} E_{0,m,n,\mathbf{a}_m}$, $\mathbf{a}_m = (a_1, \ldots, a_m)$, and $E_{0,m,n,\mathbf{a}_m}$ is a $2m$th-order differential form in $\mathbb{R}[\mathcal{P}_{m,n}]$.

We can now prove Lemma 2.1 for $s = 2, 3$. Let

$$\begin{aligned} E_{2,m,n} &= E_{1,m,n} - \tfrac{(m-1)!}{2n^{m-1}} E_{0,m,n} \\ E_{3,m,n} &= E_{1,m,n} - \tfrac{(m-1)!}{2n^{m}} E_{0,m,n} \end{aligned} \tag{15}$$

where $E_{1,m,n}$ and $E_{0,m,n}$ are from Lemmas 2.8 and 2.10. By Lemma 2.9, $C_s(m,n)$ is true if $\int_{\mathbb{R}}^{n} \frac{E_{s,m,n}}{p_t^{2m-1}} dx_t \geq 0$ for $s = 2, 3$. Together with Lemma 2.8, Lemma 2.1 is proved.

From Lemma 2.9, we can prove $C_2(1,n)$, that is

Theorem 2.11: Subject to $\mathrm{Var}(X_t) = (\sigma^2 + t) \times I$, $(-1)^{n+1} \frac{\mathrm{d}}{\mathrm{d}t} H(X_t)$ achieves the minimum when X_t is Gaussian with variance $(\sigma^2 + t) \times I$ for $t > 0$ and $n \geq 1$.

Proof: Among distributions with a fixed variance $(\sigma^2 + t) \times I$, we have $(\mathrm{d}/\mathrm{d}t) H(X_t) \overset{(11)}{=} \frac{1}{2}\mathbb{E}(-T) \overset{(13)}{\geq} \frac{1}{2}\mathbb{E}(-T_G) \overset{(11)}{=} (\mathrm{d}/\mathrm{d}t) H(X_{Gt})$, and the theorem is proved. ■

D. Proof of Lemma 2.5

A symmetric matrix $\mathcal{M} \in \mathbb{R}^{n \times n}$ is called *negative semidefinite* and is denoted as $\mathcal{M} \preceq 0$, if all its eigenvalues are nonpositive. From [20], p_t is log-concave if and only if for all $x \in \mathbb{R}^n$ and $t > 0$, $\mathbf{L}(p_t)$ in (5) is negative semidefinite. By the knowledge of linear algebra, $\mathbf{L}(p_t) \preceq 0$ if and only if

$$(-1)^k \triangle_{k,l} \geq 0 \text{ for } 1 \leq k \leq n, 1 \leq l \leq \binom{n}{k} \tag{16}$$

where $\triangle_{k,l}$ is a kth-order principle minors of $\mathbf{L}(p_t)$. Note that elements of $\mathbf{L}(p_t)$ are quadratic differential forms in $\mathbb{R}[\mathcal{P}_n]$. Then $(-1)^k \triangle_{k,l}$ is a kth-order log-concave constraint. As a consequence, $\prod_{i=1}^{s} (-1)^{k_i} \triangle_{k_i,l_i} Q_{k_1,\ldots,k_s}$ is an mth-order log-concave constraint, if $Q_{k_1,\ldots,k_s} \in \mathrm{Span}_{\mathbb{R}}(\mathcal{M}_{2m-2\sum_{i=1}^{s} k_i, n})$ and $Q_{k_1,\ldots,k_s} \succeq 0$. This proves Lemma 2.5. And an illustrative example has been presented in [19].

E. Procedure 2.6

The inputs: $E_{s,m,n}$ and $R_i, i = 1, \ldots, N_1$ are $2m$th-order differential forms in $\mathbb{R}[\mathcal{P}_n]$; $P_j, j = 1, \ldots, N_2$ are $2k_j$th-order differential forms in $\mathbb{R}[\mathcal{P}_n]$.

The outputs: $e_i \in \mathbb{R}$ and $Q_j \in \mathrm{Span}_{\mathbb{R}}(\mathcal{M}_{2(m-k_j),n})$ such that (8) and (9) are true; or fail meaning that such e_i and Q_j are not found.

S1. Treat the monomials in $\mathcal{M}_{m,n}$ as new variables $m_l, l = 1, \ldots, N_{m,n}$, which are all the monomials in $\mathbb{R}[\mathcal{P}_n]$ with degree m and total order m. We call $m_l m_s$ a *quadratic monomial*.

S2. Write monomials in $\mathcal{C}_{m,n} = \{R_i, i = 1, \ldots, N_1\}$ as quadratic monomials if possible. Doing Gaussian elimination to $\mathcal{C}_{m,n}$ by treating the monomials as bases and according to a monomial order such that a quadratic monomial is less than a non-quadratic monomial, we obtain

$$\widetilde{\mathcal{C}}_{m,n} = \mathcal{C}_{m,n,1} \cup \mathcal{C}_{m,n,2},$$

where $\mathcal{C}_{m,n,1}$ is the set of quadratic forms in m_i, $\mathcal{C}_{m,n,2}$ is the set of non-quadratic forms, and $\mathrm{Span}_{\mathbb{R}}(\mathcal{C}_{m,n}) = \mathrm{Span}_{\mathbb{R}}(\widetilde{\mathcal{C}}_{m,n})$.

S3. There may exist *intrinsic constraints*. For instance, for $m_1 = p_t^2 (\frac{\partial^2 p_t}{\partial^2 x_{1,t}})^2$, $m_2 = p_t (\frac{\partial p_t}{\partial x_{1,t}})^2 \frac{\partial^2 p_t}{\partial^2 x_{1,t}}$, and $m_3 = (\frac{\partial p_t}{\partial x_{1,t}})^4$ in $\mathcal{M}_{4,n}$, an intrinsic constraint is $m_1 m_3 - m_2^2 = 0$. Add the intrinsic constraints which are quadratic forms in m_i to $\mathcal{C}_{m,n,1}$, we obtain

$$\widehat{\mathcal{C}}_{m,n,1} = \{\widehat{R}_i, i = 1, \ldots, N_3\}.$$

S4. Let $\mathcal{M}_{2(m-k_j),n} = \{m_{j,k}, k = 1, \ldots, V_j\}$ and $Q_j = \sum_{k=1}^{V_j} q_{j,k} m_{j,k}$, where $q_{j,k} \in \mathbb{R}$ are variables to be found later. Let $\mathcal{R}_j$ be obtained from $P_j Q_j$ by writing monomials in $P_j Q_j$ as quadratic monomials in m_i and eliminating the non-quadratic monomials with $\mathcal{C}_{m,n,2}$, such that $\mathcal{R}_j - P_j Q_j \in \mathrm{Span}_{\mathbb{R}}(\mathcal{C}_{m,n})$ and $\mathcal{R}_j = \sum_{l=1}^{V_j} q_{j,l} h_{j,l}$, where $h_{j,l} \in \mathbb{R}[m_i, \mathcal{P}_n]$. If $h_{j,l}$ is not a quadratic form in m_i for some l, then set $\mathcal{R}_j = 0$ and still denote these constraints as $\widehat{\mathbf{C}}_{m,n} = \{\mathcal{R}_j, j = 1, \ldots, N_2\}$.

S5. Let $\widehat{E}_{s,m,n}$ be obtained from $E_{s,m,n}$ by eliminating the non-quadratic monomials using $\mathcal{C}_{m,n,2}$ such that $E_{s,m,n} - \widehat{E}_{s,m,n} \in \mathrm{Span}_{\mathbb{R}}(\mathcal{C}_{m,n,2}) \subset \mathrm{Span}_{\mathbb{R}}(\mathcal{C}_{m,n})$.

S6. Since $\widehat{E}_{s,m,n}$, $\widehat{R}_i, i = 1, \ldots, N_3$ and $\mathcal{R}_j, j = 1, \ldots, N_2$ are quadratic forms in m_i, we can use the Matlab software given in Appendix A of [19] to compute $p_i, q_{j,s} \in \mathbb{R}$ s.t.

$$\widehat{E}_{s,m,n} - \textstyle\sum_{i=1}^{N_3} p_i \widehat{R}_i - \sum_{j=1}^{N_2} \mathcal{R}_j = S, \tag{17}$$

$$\begin{aligned} \mathcal{R}_j &= \textstyle\sum_{l=1}^{V_j} q_{j,l} h_{j,l}, j = 1, \ldots, N_2 \\ Q_j &= \textstyle\sum_{l=1}^{V_j} q_{j,l} m_{j,l} \geq 0, j = 1, \ldots, N_2 \end{aligned} \tag{18}$$

where $S = \sum_{i=1}^{N_{m,n}} c_i (\sum_{j=i}^{N_{m,n}} e_{ij} m_j)^2$ is an SOS, $c_i, e_{ij} \in \mathbb{R}$ and $c_i \geq 0$. If (17) and (18) cannot be found, return fail.

S7. Since $\widehat{R}_i$, $E_{s,m,n} - \widehat{E}_{s,m,n}$, $\mathcal{R}_j - P_j Q_j$ are all in $\mathrm{Span}_{\mathbb{R}}(\mathcal{C}_{m,n})$, equations (8) and (9) can be obtained from (17) and (18), respectively.

III. Proof of $C_2(2,n)$

We prove $C_2(2,n)$ using the procedure given in section II-B.

Theorem 3.1: Subject to $Var(X_t) = (\sigma^2 + t) \times I$, Gaussian X_t with variance $(\sigma^2 + t) \times I$ achieves the minimum of $(-1)^{n+1} \frac{d^2}{d^2 t} H(X_t)$ for $t > 0$ and $n \geq 1$.

When $m = 2$, (13) holds without log-concave condition. Thus, in this case, the log-concave conditions are not needed, we set $Q_j = 0$ in (8).

A. Compute $E_{2,2,n}$

In step 1, we compute $E_{2,2,n}$ with (15):

$$-\frac{\mathrm{d}^2 H(X_t)}{\mathrm{d}^2 t} - \frac{1}{2n} \mathbb{E}(\frac{\|\nabla p_t\|^2 - p_t \nabla^2 p_t}{p_t^2})^2 = \int \frac{E_{2,2,n}}{p_t^3} \mathrm{d}x_t \tag{19}$$

where

$$\begin{aligned} E_{2,2,n} &= -\tfrac{\mathrm{d}}{\mathrm{d}t}(\tfrac{\|\nabla p_t\|^2}{2p_t}) - \tfrac{1}{2n} \tfrac{(\|\nabla p_t\|^2 - p_t \nabla p_t)^2}{p_t^3} \\ &= -\tfrac{1}{2} p_t^2 \nabla p_t \cdot \nabla(\nabla^2 p_t) + \tfrac{1}{4} p_t \|\nabla p_t\|^2 \nabla^2 p_t \\ &\quad - \tfrac{1}{2n} (\|\nabla p_t\|^2 - p_t \nabla^2 p_t)^2 \\ &= \sum_{a=1}^{n} \sum_{b=1}^{n} (T_{1,a,b} - \tfrac{1}{2n} T_{2,a,b}), \text{ and} \\ T_{1,a,b} &= -\tfrac{1}{2} p_t^2 \tfrac{\partial p_t}{\partial x_{a,t}} \tfrac{\partial^3 p_t}{\partial x_{a,t} \partial^2 x_{b,t}} + \tfrac{1}{4} p_t (\tfrac{\partial p_t}{\partial x_{a,t}})^2 \tfrac{\partial^2 p_t}{\partial^2 x_{b,t}} \\ T_{2,a,b} &= ((\tfrac{\partial p_t}{\partial x_{a,t}})^2 - p_t \tfrac{\partial^2 p_t}{\partial^2 x_{a,t}})((\tfrac{\partial p_t}{\partial x_{b,t}})^2 - p_t \tfrac{\partial^2 p_t}{\partial^2 x_{b,t}}) \end{aligned} \tag{20}$$

B. The second order constraints

In step 2, we compute the second order integral constraints. Due to the summation structure of $E_{2,2,n}$ in (20), we introduce the following notations

$$\mathcal{V}_{a,b} = \{\frac{\partial^h p_t}{\partial^{h_1} x_{a,t} \partial^{h_2} x_{b,t}} : h = h_1 + h_2 \in [3]_0\} \tag{21}$$

where a, b are variables taking values in $[n]$. Then $\mathcal{P}_{2,n} = \cup_{a=1}^n \cup_{b=1}^n \mathcal{V}_{a,b}$. The second order integral constraints are [19]:

$$\mathcal{C}_{2,n} = \{R_{i,a,b}^{(2)}, R_j^{(0)} \,:\, i = 1, \ldots, 17; j = 1, 2; a, b \in [n]\}. \tag{22}$$

C. Prove $C_2(2, n)$

In step 3, we use Procedure 2.6 to prove $C_2(2, n)$ with $E_{2,2,n}$ and $\mathcal{C}_{2,n}$ in (22) as input. It suffices to write

$$E_{2,2,n} - \sum_{R \in \mathcal{C}_{2,n}} c_R R = S \geq 0 \tag{23}$$

where $c_R \in \mathbb{R}$ and S is an SOS. From (23), a proof for $C_2(2, n)$ can be given based on Theorem 2.7. Since $C_2(2, 1)$ was proved in [13], [17], we will consider $C_2(2, n),\ n \geq 2$. Due to the parameters a and b, the problem cannot be proved directly with Procedure 2.6. We will reduce the problem to a "finite" problem which can be solved with Procedure 2.6.

From (19) and (22), to prove (23), it suffices to solve **Problem I**. There exist $c_1, c_2 \in \mathbb{R}$ and an SOS S such that

$$\widetilde{E}_{2,2,n} = \sum_{a=1}^n \sum_{b=1}^n (T_{1,a,b} - \tfrac{1}{2n} T_{2,a,b} + c_1 R_{1,a,b}^{(0)} + c_2 R_{2,a,b}^{(0)}) = S$$

under the constraints $R_{i,a,b}^{(2)}, i = 1, \ldots, 17$ given in (22).

Motivated by symmetric functions, for any function $f(a, b)$,

$$\sum_{a,b=1}^n f(a, b) = \sum_{1 \leq a < b}^n \left\{ \tfrac{1}{n-1} [f(a, a) + f(b, b)] + [f(a, b) + f(b, a)] \right\}. \tag{24}$$

By (24), we have

$$\begin{aligned}
\widetilde{E}_{2,2,n} &= \sum_{a=1}^n \sum_{b=1}^n (T_{1,a,b} - \tfrac{1}{2n} T_{2,a,b} + c_1 R_{1,a,b}^{(0)} + c_2 R_{2,a,b}^{(0)}) \\
&= \sum_{a<b} \Big[\tfrac{1}{n-1} (T_{1,a,a} + T_{1,b,b} - \tfrac{1}{2n} (T_{2,a,a} + T_{2,b,b}) + c_1 (R_{1,a,a}^{(0)} + R_{1,b,b}^{(0)}) \\
&\quad + c_2 (R_{2,a,a}^{(0)} + R_{2,b,b}^{(0)})) + T_{1,a,b} + T_{1,b,a} - \tfrac{1}{2n} (T_{2,a,b} + T_{2,b,a}) \\
&\quad + c_1 (R_{1,a,b}^{(0)} + R_{1,b,a}^{(0)}) + c_2 (R_{2,a,b}^{(0)} + R_{2,b,a}^{(0)}) \Big] \\
&= \sum_{a<b} \Big\{ \tfrac{1}{n-1} [(T_{1,a,a} + T_{1,b,b}) - \tfrac{1}{2} (T_{2,a,a} + T_{2,b,b}) + c_1 (R_{1,a,a}^{(0)} + R_{1,b,b}^{(0)}) \\
&\quad + c_2 (R_{2,a,a}^{(0)} + R_{2,b,b}^{(0)})] + \tfrac{1}{2n} [(T_{2,a,a} + T_{2,b,b}) - (T_{2,a,b} + T_{2,b,a})] \\
&\quad + [(T_{1,a,b} + T_{1,b,a}) + c_1 (R_{1,a,b}^{(0)} + R_{1,b,a}^{(0)}) + c_2 (R_{2,a,b}^{(0)} + R_{2,b,a}^{(0)})] \Big\} \\
&= \sum_{1 \leq a < b \leq n} \left(\tfrac{1}{n-1} L_{1,a,b} + \tfrac{1}{2n} L_{2,a,b} + L_{3,a,b} \right),
\end{aligned}$$

where

$$\begin{aligned}
L_{1,a,b} &= (T_{1,a,a} + T_{1,b,b}) - \tfrac{1}{2} (T_{2,a,a} + T_{2,b,b}) \\
&\quad + c_1 (R_{1,a,a}^{(0)} + R_{1,b,b}^{(0)}) + c_2 (R_{2,a,a}^{(0)} + R_{2,b,b}^{(0)}), \\
L_{2,a,b} &= (T_{2,a,a} + T_{2,b,b}) - (T_{2,a,b} + T_{2,b,a}), \\
L_{3,a,b} &= (T_{1,a,b} + T_{1,b,a}) + c_1 (R_{1,a,b}^{(0)} + R_{1,b,a}^{(0)}) + c_2 (R_{2,a,b}^{(0)} + R_{2,b,a}^{(0)}).
\end{aligned}$$

To prove **Problem I**, it suffices to prove

Problem II. There exist $c_1, c_2 \in \mathbb{R}$ and SOSs S_1, S_2, S_3 such that $L_{1,a,b} = S_1, L_{2,a,b} = S_2, L_{3,a,b} = S_3$ under the constraints $R_{i,a,b}^{(2)}, i = 1, \ldots, 17$.

In **Problem II**, the subscripts a and b are fixed and we can prove **Problem II** using Procedure 2.6 with $L_{1,a,b}, L_{2,a,b}, L_{3,a,b}$ and $R_{i,a,b}^{(2)}, i = 1, \ldots, 17$ as input.

Step **S1**. The new variables are all the monomials in $\mathbb{R}[\mathcal{V}_{a,b}]$ with degree 2 and total order 2 ($\mathcal{V}_{a,b}$ is defined in (21)):

$$\begin{aligned}
&m_1 = \left(\tfrac{\partial p_t(x_t)}{x_{a,t}}\right)^2,\ m_2 = \left(\tfrac{\partial p_t(x_t)}{x_{b,t}}\right)^2,\ m_3 = \tfrac{\partial p_t(x_t)}{\partial x_{a,t}} \tfrac{\partial p_t(x_t)}{x_{b,t}}, \\
&m_4 = p_t(x_t) \tfrac{\partial^2 p_t(x_t)}{\partial x_{a,t} \partial x_{b,t}},\ m_5 = p_t(x_t) \tfrac{\partial^2 p_t(x_t)}{\partial^2 x_{a,t}},\ m_6 = p_t(x_t) \tfrac{\partial^2 p_t(x_t)}{\partial^2 x_{b,t}}.
\end{aligned}$$

Step **S2**. We obtain $\mathcal{C}_{2,n,1} = \{\widehat{R}_i, i = 1, \ldots, 7\}$ and $\mathcal{C}_{2,n,2} = \{\widetilde{R}_i, i = 1, \ldots, 10\}$ using Gaussian elimination, where

$$\begin{aligned}
&\widehat{R}_1 = m_1 m_6 - 2m_3^2 + 2m_3 m_4,\ \widehat{R}_2 = -2m_2 m_3 + m_2 m_4 + 2m_3 m_6, \\
&\widehat{R}_3 = -2m_2^2 + 3m_2 m_6,\ \widehat{R}_4 = -2m_1 m_3 + m_1 m_4 + 2m_3 m_5, \\
&\widehat{R}_5 = m_2 m_5 - 2m_3^2 + 2m_3 m_4,\ \widehat{R}_6 = -2m_2 m_3 + 3m_2 m_4, \\
&\widehat{R}_7 = -2m_1^2 + 3m_1 m_5. \\
&\widetilde{R}_1 = p_t^2 \tfrac{\partial p_t}{\partial x_{b,t}} \tfrac{\partial^3 p_t}{\partial^3 x_{b,t}} - m_2 m_6 + m_6^2, \widetilde{R}_2 = p_t^2 \tfrac{\partial p_t}{\partial x_{a,t}} \tfrac{\partial^3 p_t}{\partial^3 x_{a,t}} - m_1 m_5 + m_5^2, \\
&\widetilde{R}_3 = p_t^2 \tfrac{\partial p_t}{\partial x_{a,t}} \tfrac{\partial^3 p_t}{\partial x_{a,t} \partial^2 x_{b,t}} - m_3 m_4 + m_4^2, \\
&\widetilde{R}_4 = p_t^2 \tfrac{\partial p_t}{\partial x_{b,t}} \tfrac{\partial^3 p_t}{\partial^2 x_{a,t} x_{b,t}} - m_3 m_4 + m_4^2.
\end{aligned}$$

Here, $\widetilde{R}_5, \ldots, \widetilde{R}_{10}$ are omitted, since they are not used in the proof.

Step **S3**. There exists one intrinsic constraint: $\widehat{R}_8 = m_1 m_2 - m_3^2$ and $N_3 = 8$.

Step **S4** is not needed, since there are no log-concave constraints.

Step **S5**. Eliminating the non-quadratic monomials in $L_{1,a,b}$, $L_{2,a,b}$, and $L_{3,a,b}$ using $\mathcal{C}_{2,n,2}$, and doing further reduction by $\mathcal{C}_{2,n,1}$, we have

$$\begin{aligned}
\widehat{L}_{1,a,b} &= L_{1,a,b} + (\tfrac{1}{2} - c_1) \widetilde{R}_1 + (\tfrac{1}{2} - c_1) \widetilde{R}_2 - (\tfrac{1}{4} + c_2) \widehat{R}_3 - (\tfrac{1}{4} + c_2) \widehat{R}_7 = 0, \\
\widehat{L}_{2,a,b} &= L_{2,a,b} - 2\widehat{R}_1 + \tfrac{1}{2} \widehat{R}_3 - 2\widehat{R}_5 + \tfrac{1}{2} \widehat{R}_7 \\
&= -\tfrac{1}{2} m_1 m_5 - \tfrac{1}{2} m_2 m_6 + 6m_3^2 - 8m_3 m_4 + m_5^2 - 2m_5 m_6 + m_6^2, \\
\widehat{L}_{3,a,b} &= L_{3,a,b} + (\tfrac{1}{2} - c_1) \widetilde{R}_3 + (\tfrac{1}{2} - c_1) \widetilde{R}_4 + (c_1 - c_2 - \tfrac{1}{4}) \widehat{R}_1 \\
&\quad + (c_1 - c_2 - \tfrac{1}{4}) \widehat{R}_5 \\
&= m_3^2 - 2m_3 m_4 + m_4^2 + c_1 (-4m_3^2 + 6m_3 m_4 - 2m_4^2 + 2m_5 m_6)
\end{aligned}$$

which are quadratic forms in m_i.

Step **S6**. We obtain the following SOS representation

$$\begin{aligned}
&\widehat{L}_{1,a,b} = 0,\quad \widehat{L}_{2,a,b} = \sum_{k=1}^8 p_k \widehat{R}_k + (m_1 - m_2 - m_5 + m_6)^2, \\
&\widehat{L}_{3,a,b} = (m_3 - m_4)^2,
\end{aligned} \tag{25}$$

where $p_1 = \frac{1}{2},\ p_2 = \frac{1}{2},\ p_3 = 2,\ p_6 = -2,\ p_7 = -2, c_1 = c_2 = p_4 = p_5 = p_8 = 0$. So, **Problem II** is solved and thus $C_2(2, n)$ is proved.

IV. Proof of $C_3(3, n)$ for $n = 2, 3, 4$ under the log-concave condition

We use the procedure in section II-B to prove $C_3(3, n)$ for $n = 2, 3, 4$ under the log-concave condition.

A. Compute $E_{3,3,n}$

In step 1, we compute $E_{3,3,n}$ in (3) and (15):

$$\frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}t^2} (\frac{\|\nabla p_t\|^2}{p_t}) - \frac{1}{n^3} \mathbb{E}(\frac{\|\nabla p_t\|^2 - p_t \nabla^2 p_t}{p_t^2})^3 \overset{(11)}{=} \int_{\mathbb{R}^n} \frac{E_{3,3,n}}{p_t^5} \mathrm{d}x_t \tag{26}$$

where $E_{3,3,n} = \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n E_{3,a,b,c}$ and

$$\begin{aligned}
E_{3,a,b,c} &= \tfrac{p_t^4}{4} \tfrac{\partial^3 p_t}{\partial x_{a,t} \partial^2 x_{c,t}} \tfrac{\partial^3 p_t}{\partial x_{a,t} \partial^2 x_{b,t}} - \tfrac{p_t^3}{4} \tfrac{\partial p_t}{\partial x_{a,t}} \tfrac{\partial^3 p_t}{\partial x_{a,t} \partial^2 x_{b,t}} \tfrac{\partial^2 p_t}{\partial^2 x_{c,t}} \\
&\quad + \tfrac{p_t^4}{4} \tfrac{\partial p_t}{\partial x_{a,t}} \tfrac{\partial^5 p_t}{\partial x_{a,t} \partial^2 x_{b,t} \partial^2 x_{c,t}} - \tfrac{p_t^3}{4} \tfrac{\partial p_t}{\partial x_{a,t}} \tfrac{\partial^3 p_t}{\partial x_{a,t} \partial^2 x_{c,t}} \tfrac{\partial^2 p_t}{\partial^2 x_{b,t}} \\
&\quad + \tfrac{p_t^2}{4} \left(\tfrac{\partial p_t}{\partial x_{a,t}}\right)^2 \tfrac{\partial^2 p_t}{\partial^2 x_{b,t}} \tfrac{\partial^2 p_t}{\partial^2 x_{c,t}} - \tfrac{p_t^3}{8} \left(\tfrac{\partial p_t}{\partial x_{a,t}}\right)^2 \tfrac{\partial^4 p_t}{\partial^2 x_{b,t} \partial^2 x_{c,t}} \\
&\quad - \tfrac{1}{n^3} [(\tfrac{\partial p_t}{\partial x_{a,t}})^2 - p_t (\tfrac{\partial^2 p_t}{\partial^2 x_{a,t}})][(\tfrac{\partial p_t}{\partial x_{b,t}})^2 \\
&\quad - p_t (\tfrac{\partial^2 p_t}{\partial^2 x_{b,t}})][(\tfrac{\partial p_t}{\partial x_{c,t}})^2 - p_t (\tfrac{\partial^2 p_t}{\partial^2 x_{c,t}})].
\end{aligned}$$

B. Compute the third order constraints

Similar to (21), we introduce the notation

$$\mathcal{V}_{a,b,c} = \{\frac{\partial^h p_t}{\partial^{h_1} x_{a,t} \partial^{h_2} x_{b,t} \partial^{h_3} x_{c,t}} : h = h_1 + h_2 + h_3 \in [5]_0\} \quad (27)$$

where a, b, c are variables taking values in $[n]$.

The third order integral constraints are [18]:

$$\mathcal{C}_{3,n} = \{R^{(3)}_{i,a,b,c}, : i = 1, \ldots, 955; a, b, c \in [n]\}. \quad (28)$$

Note that we do not use all the third order constraints in [18].

From Lemma 2.5, we can compute the third order log-concave constraints:

$$\mathbf{C}_{3,2} = \{\mathcal{R}_1 = -\triangle_{1,1} Q_1, \mathcal{R}_2 = -\triangle_{1,2} Q_2, \mathcal{R}_3 = \triangle_{2,1} Q_3\}, \quad (29)$$

where $Q_1, Q_2 \in \mathrm{Span}_{\mathbb{R}}(\mathcal{M}_{4,2})$ and $Q_3 \in \mathrm{Span}_{\mathbb{R}}(\mathcal{M}_{2,2})$. Note that $\mathbf{C}_{3,2}$ does not contain all the log-concave constraints in Lemma 2.5. The constraints $\mathbf{C}_{3,2}$ are enough for our purpose.

For $n > 2$, we need certain log-concave constraints in a special form. Let $\nabla_1 p_t = (\frac{\partial p_t}{\partial x_{a,t}}, \frac{\partial p_t}{\partial x_{b,t}}, \frac{\partial p_t}{\partial x_{c,t}})$, $\mathbf{L}_1(p_t) \triangleq p_t \mathbf{H}_1(p_t) - \nabla_1^T p_t \nabla_1 p_t$, where

$$\mathbf{H}_1(p_t) = \begin{bmatrix} \frac{\partial^2 p_t}{\partial^2 x_{a,t}} & \frac{\partial^2 p_t}{\partial x_{a,t} \partial x_{b,t}} & \frac{\partial^2 p_t}{\partial x_{a,t} \partial x_{c,t}} \\ \frac{\partial^2 p_t}{\partial x_{a,t} \partial x_{b,t}} & \frac{\partial^2 p_t}{\partial^2 x_{b,t}} & \frac{\partial^2 p_t}{\partial x_{b,t} \partial x_{c,t}} \\ \frac{\partial^2 p_t}{\partial x_{a,t} \partial x_{c,t}} & \frac{\partial^2 p_t}{\partial x_{b,t} \partial x_{c,t}} & \frac{\partial^2 p_t}{\partial^2 x_{c,t}} \end{bmatrix},$$

and $\triangle'_{k,l}, l = 1, \ldots, L_k$ the kth-order principle minors of $\mathbf{L}_1(p_t)$. Let $\mathcal{M}'_k$ be the set of all monomials in $\mathcal{V}_{a,b,c}$ (defined in (27)) which have degree k and total order k. We have

$$\begin{aligned} \mathbf{C}_{3,n} = \{ & -\triangle'_{1,1} Q_{1,1}, -\triangle'_{1,2} Q_{1,2}, -\triangle'_{1,3} Q_{1,3}, \\ & \triangle'_{2,1} Q_{2,1}, \triangle'_{2,2} Q_{2,2}, \triangle'_{2,3} Q_{2,3}, -\triangle'_{3,1} Q_{3,1}\} \end{aligned} \quad (30)$$

where $Q_{1,i} \in \mathrm{Span}_{\mathbb{R}}(\mathcal{M}'_4)$, $Q_{2,j} \in \mathrm{Span}_{\mathbb{R}}(\mathcal{M}'_2)$, and $Q_{3,1} \in \mathbb{R}$.

C. Proof of $C_3(3, 2)$

The proof follows Procedure 2.6 with $E_{3,3,2}$ given in (26) and the constraints in (28) and (29) as input.

In Step **S1**, the new variables are $\mathcal{M}_{3,2}$ and are listed in the lexicographical monomial order [19].

In Step **S2**, the constraints are $\mathcal{C}_{3,2} = \{R^{(3)}_{j,a,b,c} : j = 1, \ldots, 955; a, b, c \in [2]\}$. Removing the repeated ones, we have $N_1 = 135$. We obtain $\mathcal{C}_{3,2,1}$ and $\mathcal{C}_{3,2,2}$ which contain 48 and 52 constraints, respectively.

In Step **S3**, there exist 15 intrinsic constraints [19]. Thus, $\widehat{\mathcal{C}}_{3,2,1}$ contains 63 constraints and $N_3 = 63$.

In Step **S4**, we obtain $\widehat{\mathbf{C}}(3, 2)$ which contains 3 quadratic form constraints.

In Step **S5**, eliminating the non-quadratic monomials in $E_{3,3,2}$ using $\mathcal{C}_{3,2,2}$ to obtain a quadratic form in m_i and then simplifying the quadratic form using $\mathcal{C}_{3,2,1}$, we get $\widehat{E}_{3,3,2}$ [19].

In Step **S6**, using the Matlab software in Appendix A of [19] with $\widehat{E}_{3,3,2}$, $\widehat{\mathcal{C}}_{3,2,1}$ and $\widehat{\mathbf{C}}_{3,2}$ as input, we find an SOS representation for $\widehat{E}_{3,3,2}$. Thus, $C_3(3, 2)$ is proved under the log-concave condition. The program to prove $C_3(3, 2)$ can be found in https://github.com/cmyuanmmrc/codeforepi/.

Remark 4.1: We fail to prove $C_2(3, 2)$ even under the log-concave condition using the above procedure. Specifically, we cannot find an SOS representation for $\widehat{E}_{2,3,2}$ in Step **S6**. Since the SDP algorithm is not complete for problem (17), we cannot say that an SOS representation does not exist for $\widehat{E}_{2,3,2}$. The program for $C_2(3, 2)$ can be found in https://github.com/cmyuanmmrc/codeforepi/.

D. Proof of $C_3(3, 3)$ and $C_3(3, 4)$

In this subsection, we prove $C_3(3, 3), C_3(3, 4)$. Motivated by symmetric functions, we obtain

$$E_{3,3,n} = \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n E_{3,a,b,c} = \sum_{1 \le a < b < c \le n}^{n} J_{3,3,n},$$

where

$$\begin{aligned} J_{3,3,n} = & \tfrac{2}{(n-1)(n-2)} [E_{3,a,a,a} + E_{3,b,b,b} + E_{3,c,c,c}] + \tfrac{1}{n-2} [E_{3,a,a,b} \\ & + E_{3,a,b,a} + E_{3,b,a,a} + E_{3,a,a,c} + E_{3,a,c,a} + E_{3,c,a,a} + E_{3,b,b,a} \\ & + E_{3,b,a,b} + E_{3,a,b,b} + E_{3,b,b,c} + E_{3,b,c,b} + E_{3,c,b,b} + E_{3,c,c,a} \\ & + E_{3,c,a,c} + E_{3,a,c,c} + E_{3,c,c,b} + E_{3,c,b,c} + E_{3,b,c,c}] + [E_{3,a,b,c} \\ & + E_{3,a,c,b} + E_{3,b,a,c} + E_{3,b,c,a} + E_{3,c,a,b} + E_{3,c,b,a}]. \end{aligned} \quad (31)$$

From (31), if we prove $J_{3,3,n} \ge 0$, then $E_{3,3,n} \ge 0$. It is clear that $J_{3,3,n}$ has much fewer terms than $E_{3,3,n}$.

In $J_{3,3,n}$ given in (31) and the constraints in (28) and (30), we may consider $\frac{\partial}{\partial x_{a,t}}$, $\frac{\partial}{\partial x_{b,t}}$, and $\frac{\partial}{\partial x_{c,t}}$ as the differential operators without giving concrete values to a, b, c.

First, we prove of $C_3(3, 3)$ using Procedure 2.6 with $J_{3,3,3}$ given in (31) and the constraints in (28) and (30) as the input.

In Step **S1**, the new variables are $\mathcal{M}'_3 = \{m_i, i = 1, \ldots, 38\}$ which is the set of all monomials in $\mathbb{R}[\mathcal{V}_{a,b,c}]$ with degree 3 and total order 3.

In Step **S2**, the constraints are: $\mathcal{C}_{3,n} = \{R^{(3)}_{i,a,b,c} : i = 1, \ldots, 955\}$, $N_1 = 955$. We obtain $\mathcal{C}_{3,n,1}$ and $\mathcal{C}_{3,n,2}$, which contain 350 and 328 constraints, respectively.

In Step **S3**, there exist 189 intrinsic constraints. In total, $\widehat{\mathcal{C}}_{3,n,1}$ contains 539 constraints. Using $\mathbb{R}$-Gaussian elimination in $\mathrm{Span}_{\mathbb{R}}(\widehat{\mathcal{C}}_{3,n,1})$ shows that 512 of these 539 constraints are linearly independent, so $N_3 = 512$.

In Step **S4**, we obtain $\widehat{\mathbf{C}}_{3,n}$ from $\mathbf{C}_{3,n}$ which contains 6 constraints.

In Step **S5**, eliminating the non-quadratic monomials in $J_{3,3,3}$ using $\mathcal{C}_{3,n,2}$ and then simplify the expression using $\mathcal{C}_{3,n,1}$, we obtain $\widehat{J}_{3,3,3}$.

In Step **S6**, using the Matlab software in Appendix A of [19] with $\widehat{J}_{3,3,3}$, $\widehat{\mathcal{C}}_{3,n,1}$ and $\widehat{\mathbf{C}}_{3,n}$ as input, we find an SOS representation for $\widehat{J}_{3,3,3}$. Thus, $C_3(3, 3)$ is proved. The program to prove $C_3(3, 3)$ can be found in https://github.com/cmyuanmmrc/codeforepi/.

To prove $C_3(3, 4)$, we just need to replace the input from $J_{3,3,3}$ to $J_{3,3,4}$ in the Step **S5** in the above procedure. In the same way, $C_3(3, 4)$ can be proved. The program to prove $C_3(3, 4)$ can be found in https://github.com/cmyuanmmrc/codeforepi/.

V. Conclusion

In this paper, the lower bound for the derivatives of $H(X_t)$ are considered. We first consider a conjecture $C_2(m, n)$ of McKean in the multivariate case. We propose a general procedure to prove inequities similar to $C_2(m, n)$. Using the procedure, we prove $C_2(1, n)$, $C_2(2, n)$ (new result) and notice that $C_2(m, n)$ cannot be proved for $m > 2$ and $n > 1$ with the procedure, which motivates us to propose the a weaker conjecture $C_3(m, n)$. Using our procedure, we prove $C_3(3, 2), C_3(3, 3)$, and $C_3(3, 4)$ under the log-concave condition.

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